SOLUTIONS OF EXERCISES.

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25

An elastic ring of radius a is placed gently on a smooth paraboloid of revolution whose axis is vertical; find, by use of the principle of energy, the lowest position to which the ring will descend, and its position of static equilibrium.

[*R. D. Bohannan.*]

SOLUTION.

The forces acting on the ring are gravity, its elastic tension and the normal resistance of the surface. In the descent of the ring from its initial position to its lowest position this latter force does no work, while the work done by gravity must equal the work done by the contractile force of the ring.

Let the contractile force of the ring be P and the ring-tension T, then

$$P = 2\pi T.$$

If x be the radius of the ring in its lowest position, then

$$T = \frac{\mu}{a}(x - a),$$

μ being the modulus of elasticity of the ring.

Equating gravity work to that of the ring-tension we have, if $x^2 = my$ be the meridian of the surface, and W the weight of the ring,

$$\frac{W}{m}(x^2 - a^2) = 2\pi \int_a^x \frac{\mu}{a}(x - a)\, dx.$$

Whence

$$x = \frac{1 + \dfrac{aW}{m\pi\mu}}{1 - \dfrac{aW}{m\pi\mu}}\, a.$$

Otherwise resolving gravity and P into the tangent cone of semi-angle α, we have for the force F urging the ring along the surface,

$$F = W \cos \alpha - P \sin \alpha.$$

The energy of the ring in its lowest position is equal to the work done in bringing it there, or

$$E = \tfrac{1}{2}Mv^2 = \int_a^x F ds = 0$$

$$= \int_a^x (Wdy - Pdx),$$

since $\cos a = dy/ds$ and $\sin a = dx/ds$, and the result is the same as above.

The condition for static equilibrium is

$$W \cos a - P \sin a = 0$$

or

$$W = P \tan a.$$

$\tan a = dx/dy$, from $x^2 = my$, is $m/2x$. Hence if x_0 be the radius of the ring in equilibrium

$$x_0 = \frac{a}{1 - \dfrac{aW}{m\pi\mu}}.$$

[*W. H. Echols.*]

131

Find the most general curve of the fourth degree which shall consist of two equal symmetrical loops and have no other branch, thus forming approximately a figure eight. Discuss the changes which result from causing the constants to vary.

SOLUTION.

The conditions of the problem make it easy to choose our axes so that the most general curve shall be represented by

$$Ax^4 + By^4 + Cx^2y^2 + Dx^2 + Ey^2 = 0,$$

in which, choosing B as unity, there remain four constants to consider. We introduce the dimensions in the figure as follows:—

First, if b be the length of the intercepts on the y axis, then will $E = -b^2$. Calling m the positive slope of the tangents at origin, $D = m^2b^2$.

The equation now becomes

$$Ax^4 + y^4 + Cx^2y^2 + b^2(m^2x^2 - y^2) = 0.$$

Next, introducing the condition that $x = a$ is a tangent to the curve at the points $(a, \rho a)$ and $(a, -\rho a)$, we have for these points $dx/dy = 0$; whence, $C = w^2 - 2\rho^2$, where $w = b/a$. Introducing the given values of the co-ordinates and coefficients, we get from the equation to the curve $A = \rho^4 - w^2m^2$.

We may now write

$$(\rho^4 - m^2w^2)\, x^4 + y^4 + (w^2 - 2\rho^2)\, x^2y^2 + b^2\, (m^2x^2 - y^2) = 0.$$

It is to be remarked that w, ρ, and m are in descending order of magnitude. To determine their limitations and relations, we observe that the curve must lie entirely within $y^2 = m^2x^2$ and $y^2 - b^2 = 0$. Now, in order that the curve may not intersect the axis of x except at the origin, $A = \rho^4 - m^2w^2$ must be positive, that is $\rho^2 > mw$. Again, that $y^2 - b^2 = 0$ may not cut the curve $\rho^2 < \frac{1}{2}(m^2 + w^2)$.

Putting $\rho^4 = m^2w^2 + c^4$, $2\rho^2 = m^2 + w^2 - d^2$, where c and d are real, and reducing, we have

$$(y^2 - b^2)\,(y^2 - m^2x^2) + x^2\,(c^4x^2 + d^2y^2) = 0,$$

a curve passing through the intersections of $x^2 = 0$ and the imaginary ellipse $c^4x^2 + d^2y^2 = 0$ with the four lines $(y^2 - b^2)\,(y^2 - m^2x^2) = 0$.

To best discuss the limitations to be put upon c and d, proceed as follows: Let $\rho^2 = mw + k^2$, where k is a third real magnitude. Combining this with $2\rho^2 = m^2 + w^2 - d^2$, we obtain

$$2k^2 + d^2 = (m - w)^2,$$

and

$$c^4 = k^2\,(k^2 + 2mw).$$

Suppose m given and $w = m + \Delta$, where Δ is a positive increment; then, finally,

$$2k^2 + d^2 = \Delta^2.$$

If Δ be assigned, it is easily seen how the curve changes. Since Δ may have all values from 0 to $+\infty$, so may c and d. But when $c = 0$, there are points at infinity. We remark the following cases:—

(a) When $\Delta = 0$, so are c and d, and the curve reduces to

$$(y^2 - b^2)\,(y^2 - m^2x^2) = 0.$$

(b) When Δ is infinite, the case $c^4 = \infty$ gives four lines $x^4 = 0$; and d^2 infinite gives in the same event $x^2y^2 = 0$.

It is interesting to notice the cases when $m = 0$. The two lobes are then tangent.

When $c = 1$, and $d = \sqrt{2}$, we have two circles,

$$(x^2 + y^2)^2 = b^2y^2.$$

And in this case, if $c = 0$, we obtain the limiting curve

$$y^2\,(y^2 + d^2x^2 - b^2) = 0,$$

two straight lines and an ellipse. [*Percey F. Smith.*]

210

The length of a bar having the temperature t_o is l_o. Prove that when the bar rises to the temperature t, the length becomes

$$l = l_o e^{\varepsilon(t - t_o)},$$

e being the Napierian base of logarithms, and ε the linear coefficient of expansion, assumed constant. If ε is not constant, but a function of t, specify the conditions under which

$$l = l_o e^{[a(t-t_o)+b(t-t_o)^2+\ldots]}$$

and

$$l = l_o\,[1 + A\,(t - t_o) + B\,(t - t_o)^2 + \ldots]$$

hold true, a, b, . . . A, B, . . . being constants.

[*R. S. Woodward.*]

SOLUTION.

The conditions of the problem give

$$dl = l\varepsilon dt;$$

whence

$$\int_{l_o}^{l} \frac{dl}{l} = \varepsilon \int_{t_o}^{t} dt, \qquad \text{or} \qquad l = l_o e^{\varepsilon(t-t_o)}.$$

Now let $\varepsilon = f'(t)$. Then, as above, $l = l_o e^{f(t)-f(t_o)}$, and for the first case

$$f(t) - f(t_o) = a\,(t - t_o) + b\,(t - t_o)^2 + \ldots.$$

Differentiating, we find

$$f'(t) = \varepsilon = a + 2b\,(t - t_o) + 3c\,(t - t_o)^2 + \ldots.$$

For the second case,

$$f(t) - f(t_o) = \log_e\,[1 + A\,(t - t_o) + B\,(t - t_o)^2 + \ldots],$$

from which

$$f'(t) = \varepsilon = \frac{A + 2B\,(t - t_o) + 3C\,(t - t_o)^2 + \ldots}{1 + A\,(t - t_o) + B\,(t - t_o)^2 + \ldots}.$$

[*L. G. Weld.*]

246

Find the average distance of a given point in the surface of a circle from the circumference.

[*Artemas Martin.*]

SOLUTION.

Let P be the given point, O the centre of the circle. Let Q be any point in the circumference; join P and Q, and draw the radius OQ.

Put $OP = a$, $OQ = r$, and $\angle POQ = \varphi$.

Then $PQ = \sqrt{(a^2 + r^2 - 2ar\cos\varphi)}$, and the average length of PQ is

$$\Delta = \int_0^\pi \sqrt{(a^2 + r^2 - 2ar\cos\varphi)}\, d\varphi \div \int_0^\pi d\varphi$$

$$= \frac{1}{\pi}\int_0^\pi \sqrt{(a^2 + r^2 - 2ar\cos\varphi)}\, d\varphi.$$

Let $\varphi = \pi - 2\theta$, then $\cos\varphi = -\cos 2\theta = 2\sin^2\theta - 1$, and $d\varphi = -2d\theta$.

Substituting we have

$$\Delta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{(a^2 + r^2 + 2ar - 2ar\sin^2\theta)}\, d\theta$$

$$= \frac{2(a+r)}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{\left[1 - \frac{2ar}{(a+r)^2}\sin^2\theta\right]}\, d\theta$$

$$= \frac{2(a+r)}{\pi}\,\mathrm{E}\left[\frac{\sqrt{(2ar)}}{a+r}, \tfrac{1}{2}\pi\right].$$

[*Artemas Martin.*]

284

If the angular elevation of the summits of two spires (which appear in a straight line) is α, and the angular depressions of their reflections in a lake h feet below the point of observation are β and γ, then the horizontal distance between the spires is

$$2h\cos^2\alpha\sin(\beta - \gamma)\operatorname{cosec}(\beta - \alpha)\operatorname{cosec}(\gamma - \alpha).$$

SOLUTION.

Let B, C be the two spires, A the point of observation, A' the image of A in the lake; then in the triangle $BA'A$, $\angle B = \beta - \alpha$, $\angle A = \alpha + \frac{1}{2}\pi$; and in the triangle $CA'B$, $\angle C = \gamma - \alpha$, $\angle A' = \beta - \alpha$; and $AA' = 2h$. By the proportionality of the sides of any triangle to the sines of the opposite angles,

$$A'B = 2h\frac{\cos\alpha}{\sin(\beta - \alpha)}, \qquad BC = A'B\frac{\sin(\beta - \gamma)}{\sin(\gamma - \alpha)}, \qquad BD = BC\cos\alpha.$$

Hence the required distance BD is

$$2h\cos^2\alpha\sin(\beta - \gamma)\operatorname{cosec}(\beta - \alpha)\operatorname{cosec}(\gamma - \alpha).$$

[*R. B. Goodman.*]

285

Let a, b, c be the sides of a plane triangle, and α, β, γ the lines bisecting the angles and terminating in the opposite sides, then will

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)}{(a+b)(b+c)(c+a)} \times \text{area of triangle}.$$

SOLUTION.

Let the bisector from A cut BC in D. Then

$$AD = CD \sin C \operatorname{cosec} \tfrac{1}{2}A ;$$

$$\alpha = \frac{ab}{b+c} \sin C \operatorname{cosec} \tfrac{1}{2}A.$$

Taking similar expressions for β and γ and multiplying, we have

$$\alpha\beta\gamma = \frac{a^2b^2c^2}{(a+b)(b+c)(c+a)} \sin A \sin B \sin C \operatorname{cosec} \tfrac{1}{2}A \operatorname{cosec} \tfrac{1}{2}B \operatorname{cosec} \tfrac{1}{2}C.$$

Substituting the values of $\operatorname{cosec} \tfrac{1}{2}A$, etc., and remembering that

$$K = \text{area of triangle} = 2R^2 \sin A \sin B \sin C,$$

we have, after reduction,

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)K}{(a+b)(b+c)(c+a)}.$$

[*T. U. Taylor.*]

306

A body at distance r from the sun is moving with velocity v. Prove that the major axis of the orbit described is parallel to the direction of motion if, and only if, the velocity is "circular velocity for the distance r."

[*Ellery W. Davis.*]

SOLUTION.

The square of the velocity in the elliptical orbit, semi-major axis a, strength of centre μ, radius vector r, is

$$v^2 = \frac{\mu(2a - r)}{ar}. \tag{1}$$

When the body moves parallel to the major axis, it is at one of the extremities of the minor axis, where $r = a$, and then, and only then, (1) gives $v = \sqrt{\mu/a}$, the velocity in a circle, radius a, about a force in the centre equal to μ/a^2.

[*William Hoover.*]

307

If a horizontal beam of length $2a$ is supported at each end, and has a load in the form of an isosceles triangle, base $2a$, height b, a unit's thickness throughout, and heaviness unity; show that the deflection of the beam due to this triangular load is $\dfrac{2a^4b}{15EI}$.

[*T. U. Taylor.*]

SOLUTION.

Take the origin at one of the supports, and let the axis of x be horizontal, that of y vertical. The moment at any point $(x, 0)$ will be

$$\frac{abx}{2} - \frac{bx^2}{2a} \cdot \frac{x}{3} = EI\frac{d^2y}{dx^2}.$$

Whence, by integration,

$$EI\frac{dy}{dx} = \frac{ab}{2} \cdot \frac{x^2}{2} - \frac{b}{6a} \cdot \frac{x^4}{4} + C. \qquad (1)$$

If we put $x = a$, which makes $\frac{dy}{dx} = 0$, we have

$$C = -\frac{5a^3b}{24}.$$

Replacing C in (1) and integrating, since $y = 0$ for $x = 0$, we have

$$EIy = \frac{ab}{2} \cdot \frac{x^3}{6} - \frac{b}{6a} \cdot \frac{x^5}{20} - \frac{5a^3b}{24}x. \qquad (2)$$

The equation (2) gives the ordinate of deflection for any value of the abscissa. At the middle we have $x = a$;

$$\therefore\ y = -\frac{2a^4b}{15EI},$$

the minus sign showing that the beam is bent downward.

[*C. L. De Mott.*]

308

Let points represent complex quantities in the usual way. Show that the quartic whose zeros are any four cotangential points on a fixed circular cubic, has a fixed Jacobian. [*F. Morley.*]

SOLUTION.

Let a, b, c, d be the points. Let $\overline{bc}, \overline{ad}$ meet at l; $\overline{ca}, \overline{bd}$ at m; $\overline{ab}, \overline{cd}$ at n. Let $p = (bc - ad)/(b + c - a - d)$, etc. Then we know (see Russell, London Math. Soc. xix) that the circles c, a, n; b, d, n; a, b, m; c, d, m meet at the point p.

Consider now a circular cubic of which a, b, c, d are cotangential points. Then l, m, n are also on the cubic. Since the circle c, a, n and the line b, d, m make up one circular cubic, and the circle b, d, n and the line a, b, m make up another, the point p is on the given cubic, and is the residual of a, b, c, d, m, n and the circular points at infinity.

Let o be the point on the cubic residual with m, n; and let ∞ be the real point at infinity. Then (see Salmon, Curves, Third Edition, p. 139) we may determine p as follows: Let l' be the residual of l, l; k that of o, ∞. Then p is the residual of l', k.

Now o and l' are cotangential (Salmon, p. 132). Hence, joining them to k, ∞ and p are cotangential.

Thus the centres of the 3 pairs of Jacobian points are the points cotangential with the real point at infinity. These are marked S, U, V in the figure in Salmon, p. 249. The Jacobian points are the geometric means of each pair of these points with regard to the third. See Russell, p. 60.

[*F. Morley.*]

309

A HORIZONTAL beam of length $2a$, supported at each end, has a load in the form of an inverted parabola symmetrical with respect to the vertical line through the centre of beam. If the vertex of the parabola is b above beam, and if the load is a unit's thickness and has a heaviness unity, the deflection of the beam due to the parabolic load is $\dfrac{61a^4b}{360EI}$. [*T. U. Taylor.*]

SOLUTION.

The equations for solution are (with the usual notation)

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = w, \qquad EI\frac{d^2\delta}{dx^2} = M;$$

in which $w = b - px^2$, if $px^2 = y$ be the equation to the parabola referred to tangent at vertex and vertical there as axes.

$$\begin{aligned} M &= \int\int w\,dx\,dx \\ &= \int\int (b - px^2)\,dx\,dx \\ &= \int (bx - \tfrac{1}{3}px^3)\,dx, \end{aligned}$$

the constant being zero since F and x are zero together;

$$\therefore M = \tfrac{1}{2}bx^2 - \tfrac{1}{12}px^4 - \tfrac{1}{2}ba^2 + \tfrac{1}{12}pa^4,$$

the constant being determined by putting $M = 0$, $x = a$.

$$\begin{aligned} EI\delta &= \int\int M\,dx\,dx \\ &= \int (\tfrac{1}{6}bx^3 - \tfrac{1}{60}px^5 - \tfrac{1}{2}ba^2x + \tfrac{1}{12}pa^4x)\,dx, \end{aligned}$$

the constant being zero since $d\delta/dx = 0$ for $x = 0$;

$$\therefore EI\delta = \tfrac{1}{24}bx^4 - \tfrac{1}{360}px^6 - \tfrac{1}{4}ba^2x^2 + \tfrac{1}{24}pa^4x^2 + C.$$

$x=0$ gives δ for mid-span

$$EI\delta = C,$$

and since $\delta = 0$ for $x = a$ we have at mid-span

$$EI\delta = -\tfrac{1}{24}ba^4 + \tfrac{1}{360}pa^6 + \tfrac{1}{4}ba^4 - \tfrac{1}{24}pa^6.$$

If the parabola passes through the ends of the beam, as it is evidently intended that it should in the exercise, we have $p = b/a^2$. Whence, as required, $EI\delta = \frac{61}{360}a^4b$. [*W. H. Echols.*]

EXERCISES.

310

SOLVE for $x_1 : x_2 : x_3$ the continued equality

$$(x_2 + x_3)(x_2 - x_3)^2 = (x_3 + x_1)(x_3 - x_1)^2 = (x_1 + x_2)(x_1 - x_2)^2.$$

[*E. H. Moore.*]

311

TO FIND four biquadrate numbers whose sum is a square number.

[*Artemas Martin.*]

312*

GIVEN

$$u = x \sin x + \cos x,$$
$$v = \sin x - x \cos x;$$

to compute

$$\int \frac{x^2 dx}{u^2}, \qquad \int \frac{x^2 dx}{v^2}, \qquad \int \frac{bx^2 dx}{(au + bv)^2}.$$

[*C. Hermite.*]

313

GIVEN three points and three straight lines in a plane, the determinant of the nine perpendiculars from the points to the lines is equal to twice the product of the areas of the triangles formed by the points and by the lines, divided by the radius of the circle circumscribing the latter. [*W. W. Johnson.*]

314

A CYLINDER, diameter $2b$, intersects a sphere, diameter $2a$, the surface of the cylinder passing through the centre of the sphere. Required the part of the volume of the sphere contained by the cylinder. [*Artemas Martin.*]

* Suggested by *W. W. Beman.* Hermite (Cours d'Analyse, 1873, p. 260) says: "On n'a aucun procedé pour trouver directement."